International Journal of Theoretical Physics, Vol. 35, No. 11, 1996

Axiomatization of Quantum Logics

Sylvia Pulmannová[1]

Received June 11, 1996

Starting with a quantum logic (a σ-orthomodular poset) L, a set of probabilistically motivated axioms is suggested to identify L with a standard quantum logic $L(H)$ of all closed linear subspaces of a complex, separable, infinite-dimensional Hilbert space. Attention is paid to recent results in this field.

In the framework of the axiomatic approach known as the "quantum logic approach" it is usually assumed that a "quantum logic," that is, a mathematical representation of the set of all experimentally verifiable propositions about a physical system (equivalently, the set of all random events of a physical experiment), is a σ-orthomodular poset with a full set of states (i.e., generalized probability measures). Let us introduce the corresponding definitions.

Definition 1. A *σ-orthomodular poset* (σ-OMP) is a partially ordered set $(L, \leq)$ with a smallest element 0 and a greatest element 1 with the following properties:

(1) L carries a bijective map $a \mapsto a'$ such that for every $a, b \in L$, $a'' = a$, $a \leq b \Rightarrow a' \geq b'$, $a \vee a' = 1$, $a \wedge a' = 0$ (in the sense that the join $a \vee a'$ and the meet $a \wedge a'$ both exist and have the value indicated).

(2) Given any (finite or countably infinite) sequence $(a_i) \subset L$, $a_i \leq a_j'$, whenever $i \neq j$, the join $\vee a_i$ exists in L.

(3) L is *orthomodular*: $a \leq b \Rightarrow b = a \vee (b \wedge a')$.

If property (2) holds only for finite sequences, we say that L is an *orthomodular poset* (OMP). Two elements a, b in an OMP L are called

[1] Mathematical Institute, Slovak Academy of Sciences, 814 73 Bratislava, Slovakia.

0020-7748/96/1100-2309$09.50/0

orthogonal (written $a \perp b$) if $a \leq b'$. A family (a_i) is *orthogonal* if $a_i \perp a_j$ whenever $i \neq j$.

Definition 2. A (σ-additive) *state* on a σ-OMP L is a map $m: L \to [0, 1]$ such that $m(1) = 1$, and for any orthogonal sequence (a_i), $m(\vee a_i) = \Sigma\, m(a_i)$.

Let $\mathcal{S}$ be a family of states on L. We say that $\mathcal{S}$ is *full* if $m(a) \leq m(b)$ $\forall m \in \mathcal{S} \Rightarrow a \leq b$. We say that $\mathcal{S}$ is *σ-convex* if for any sequence $(m_i) \subset \mathcal{S}$ and any corresponding sequence t_i of nonnegative real numbers of sum one, $m := \Sigma\, t_i m_i \in \mathcal{S}$. The state m is called a convex combination (or a mixture) of the states (m_i) with weights (t_i). A state is *pure* if it is not a convex combination of two states different from itself.

Mackey (1963) proposes a probabilistically motivated axiom system for nonrelativistic quantum mechanics. Let us briefly recall his well-known axioms.

The basic elements are two abstract sets $\mathcal{O}$ and $\mathcal{S}$ and the family of Borel subsets of the real numbers. The axioms are expressed in terms of a postulated function $p(A, \alpha, X) = s \in [0, 1]$ that assigns a real number to each triple $A \in \mathcal{O}$, $\alpha \in \mathcal{S}$, and a Borel set X. The elements $A \in \mathcal{O}$ are to be thought of as observables (physical quantities), the $\alpha \in \mathcal{S}$ as states, and the real number $p(A, \alpha, X)$ as the probability that a measurement of the observable A on a system in the state α yields a value in the Borel set X. The first six axioms run as follows.

Axiom I. For any observable A and any state α, $p(A, \alpha, \emptyset) = 0$, $p(A, \alpha, R) = 1$, and $p(A, \alpha, .)$ is countably additive on the Borel sets. Thus, with A and α fixed, $p(A, \alpha, .)$ is a probability measure on the Borel sets.

Axiom II. If $p(A, \alpha, X) = p(B, \alpha, X)$ for all states α and all Borel sets X, then $A = B$. Likewise, if $p(A, \alpha, X) = p(A, \beta, X)$ for all observables A and all X, then $\alpha = \beta$.

Axiom III. For any observable A and any Borel measurable function f, there is another observable B such that $p(B, \alpha, X) = p(A, \alpha, f^{-1}(X))$ for every state α and every Borel set X.

Axiom II shows that B in Axiom III is uniquely defined by A; we set $B = f(A)$. Thus this axiom provides for the construction of functions of observables. Axiom III postulates that the set $\mathcal{O}$ of observable is closed under a functional calculus based on Borel functions. A particular role is played by characteristic functions χ_F of Borel sets F. Mackey calls these observables "questions." Another characterization of questions is as follows: an observable A is a question if for every state $\alpha \in \mathcal{S}$, its associated probability measure $p(A, \alpha, .)$ equals 1 on the two-element set $\{0, 1\}$. Thus, a measurement of a question Q in any state α yields either a value 1 or 0; the probability that the measured value will be 1 is $s = p(Q, \alpha, \{1\})$; the probability that the result of measurement will be 0 is $1 - s$.

Axiom IV. If $\alpha_1, \alpha_2, \ldots$ is a finite or countably infinite set of members of $\mathcal{S}$ and t_i is a corresponding set of positive real numbers of sum 1, then there is a state $\alpha \in \mathcal{S}$ such that $p(A, \alpha\, X) = \Sigma\, t_i p(A, \alpha, X)$ for all observables $A \in \mathbb{O}$ and all Borel sets X.

Axiom IV implies that the set $\mathcal{S}$ of states is σ-convex. The existence of pure states is not postulated.

We denote the set of questions by $\mathcal{L}$, we have $\mathcal{L} \subset \mathbb{O}$. The question $Q = \chi_F(A)$ depends on the observable A and on the Borel set F, $Q = Q(A, F)$. Every observable A can be described by the set of questions $Q(A, F)$, F-Borel set, which correspond to the question: "Is the value of the observable A in the set F?" Two special questions are $I = Q(A, \mathbb{R})$ and $O = Q(A, \emptyset)$; both are independent of the observable A. If α is any state, we define a function m_α: $\mathcal{L} \to [0, 1]$ by $m_\alpha(Q) = p(Q, \alpha, \{1\})$. Thus $m_\alpha(Q)$ is the probability that a measurement of the question Q on the system in state α yields the value 1. Axiom II implies these two facts: if $m_\alpha(Q_1) = m_\alpha(Q_2)$ for all states α, then $Q_1 = Q_2$; and if $m_\alpha(Q) = m_\beta(Q)$ for all questions Q, then $\alpha = \beta$. Hence the map $\alpha \mapsto m_\alpha$ is one-to-one, and we may identify the set $\mathcal{S}$ of states with the family of functions m_α. We shall denote this set of functions also by $\mathcal{S}$. So we have singled out a special family $\mathcal{L}$ of observables, the questions, and have identified the states $\mathcal{S}$ with certain [0, 1]-valued functions on these observables. Attention now focuses on the system $(\mathcal{L}, \mathcal{S})$. The functions m_α: $\mathcal{L} \to [0, 1]$ are used to give $\mathcal{L}$ the structure of a partially ordered set: $Q_1 \leq Q_2$ iff $m_\alpha(Q_1) \leq m_\alpha(Q_2)$ for every α. Questions I and O play the role of the greatest and the smallest element, respectively.

Axiom V. Let (Q_i) be a sequence of questions such that $m_\alpha(Q_i) + m_\alpha(Q_j) \leq 1$ whenever $i \neq j$ for every $\alpha \in \mathcal{S}$. Then there is a question Q such that, for every $\alpha \in \mathcal{S}$, $m_\alpha(Q) = \Sigma\, m_\alpha(Q_i)$.

Axiom VI. For any $Q \neq O$ there is $\alpha \in \mathcal{S}$ such that $m_\alpha(Q) = 1$.

From Axioms I–V the following algebraic structure can be derived: the set $\mathcal{L}$ is a σ-orthomodular poset, and $\mathcal{S}$ is a full set of states on $\mathcal{L}$ (Beltrametti and Cassinelli, 1981; Pták and Pulmannová, 1991; and references therein). Axiom VI implies that $\mathcal{S}$ is *unital*, in addition.

Mackey's Axiom VII reads as follows:

Axiom VII. The partially ordered set of all questions in quantum mechanics is isomorphic to the partially ordered set of all closed subspaces of a separable, infinite-dimensional Hilbert space.

Mackey introduced this axiom because "Almost all modern quantum mechanics is based implicitly or explicitly on the following assumption which we shall state as an axiom (Axiom VII)," and he comments further, "This axiom has a rather different character from axioms I through VI. These had some degree of physical naturalness and plausibility. Axiom VII seems entirely ad hoc. . . Ideally one would like to have a list of physically plausible assumptions from which one could deduce axiom VII."

Since then, many attempts have been undertaken to find simple and physically plausible assumptions which replace Axiom VII and lead naturally to a characterization of the Hilbert-space-derived logics (see, e.g., Birkhoff and von Neumann, 1936; Zieler, 1961, 1966; Piron, 1964; 1976; Amemiya and Araki, 1966/67; MacLaren, 1964; Varadarajan, 1968; Gudder and Piron, 1971; Maczyński, 1973; Cirelli and Cotta-Ramusino, 1973; Bugajska and Bugaski, 1972; Wilbur, 1977). Usually, this may be viewed as a two-stage process. First, as suggested by the original work by Birkhoff and von Neumann (1936), there are hypotheses relating the structure of the logic to the finite-dimensional projective geometries. Thus we obtain a projective logic which can be coordinatized. Then assumptions must be found which relate the coordinatizing division ring to one of the "classical" fields, namely the real field R, the complex field C, or the quaterionic field H. It was long believed that these three are the only possible examples of coordinatizing rings for projective logics, but a nonclassical example of an "orthomodular space" was found by Keller (1980).

Recently, two papers appeared suggesting a list of axioms replacing Mackey's Axiom VII (Pulmannová, 1994; Holland, 1995). Although these two papers are completely independent, the axioms are in essentials the same. This coincidence is not accidental, it is rather a confirmation that the choice of axioms is at present the best possible. The only exception is that in Holland's paper, a very recent result by Solèr (1995) is used, which replaces Wilbur's notion of a probabilistic projective logic used in Pulmannová (1994). Namely, Solèr has recently proved that an orthomodular space that has an infinite orthonormal sequence is a real, complex, or quaternionic Hilbert space.

We will now introduce a list of axioms which make a couple (L, $\mathcal{S}$), where L is a σ-orthomodular poset and $\mathcal{S}$ is a σ-convex full set of states (a situation to which we can arrive using Axioms I–V), a projective logic (or an orthomodular space), to which Piron's (1964) coordinatization theorem can be applied. Although in Mackey's axioms the existence of pure states is not explicitly required, pure states play an important role in quantum mechanics, and hence we will assume that there is a set P of pure states (extreme points of $\mathcal{S}$), subject to the following requirements. We follow essentially Pulmannová (1994).

Our first axiom specifies the dimension of our system.

A1 (Separability). Every subset K of L consisting of mutually orthogonal nonzero elements is at most countable, and there exists at least one such infinite set.

The second axiom is a strengthening of Axiom VI.

A2 (Unitality of P). For each $b \in L$, $b \neq 0$, there exists $s \in P$ such that $s(b) = 1$.

The third axiom concerns the so-called Jauch–Piron property, which was first formulated by Zierler (1961) in a lattice form, and for posets in (Bugajska and Bugajski, 1972). It was also required by Jauch and Piron as a natural property of quantum states (Jauch, 1968; Piron, 1976; Jauch and Piron, 1969; see also Rüttimann, 1977; Pták and Pulmannová, 1991).

A3 (Jauch–Piron Property). For each $s \in \mathcal{S}$ and a, $b \in L$ satisfying $s(a) = s(b) = 1$ there exists $c \in L$ such that $c \leq a$, $c \leq b$, and $s(c) = 1$.

The following axiom is in agreement with the assumption that pure states provide us maximal information about a physical system, and this information is given by the set of all "almost sure" events in this state.

A4 (Characterization of Pure States by Almost Sure Events). For any p, $s \in P$, $p^{-1}(\{1\}) \subset s^{-1}(\{1\}) \Rightarrow p = s$.

The following axiom is a formulation of a superposition principle, which is a specific property of quantum systems distinguishing them from classical ones. Its importance was emphasized by Dirac (1980). In the traditional Hilbert space approach to quantum mechanics, the formulation of the superposition principle is based on the linear structure of the state space. Namely, pure states are represented by unit vectors in a Hilbert space, and the superposition principle corresponds to the fact that any normalized linear combination of vectors represents a pure state. Since in our case there is no *a priori* linear structure of the set $\mathcal{S}$, we will use Varadarajan's (1968) definition of a superposition of states. A state s_0 is called a superposition of states in a set $Q \subset \mathcal{S}$ if $s(a) = 1$ ($a \in L$) for all $s \in Q$ implies $s_0(a) = 1$. We will be interested only in pure superpositions of pure states: let $S(a) = 1$ ($S \subset P$, $a \in L$) mean that $s(a) = 1$ for all $s \in S$, and define $\overline{S} = \{r \in P: S(a) = 1 \Rightarrow r(a) = 1\}$. That is, $\overline{S}$ is the set of all pure superpositions of S. This formulation agrees, in the Hilbert space approach, with the usual notion of a superposition. Our formulation of the superposition principle is the following (Ptàk and Pulmannová, 1991).

A5 (Superposition Principle). To every p, $q \in P$, $p \neq q$, there is $r \in \overline{\{p, q\}}$, $r \neq p$, $r \neq q$.

Our next axiom implies a covering property. The following simple form will be sufficient.

A6 (Atomic Exchange Property). If p, q, r are mutually different elements of P and $p \in \overline{\{q, r\}}$, then $q \in \overline{\{p, r\}}$.

Under the assumptions A1–A6, L is a complete, irreducible, atomic orthomodular lattice with covering property [see Pulmannová (1994) for the proof]. Moreover, for every $s \in \mathcal{S}$, the set $s^{-1}\{1\} = \{a \in L: s(a) = 1\}$ has an infimum a_s in L; the element a_s is called a carrier (or a support) of the state s. It follows that Piron's coordinatization theorem can be applied. Namely, there exist a division ring D with an involutive antiautomorphism θ, a (left) vector space V over D, and a definite θ bilinear Hermitian form $\langle ., . \rangle$ on $V \times V$ such that L is orthoisomorphic with the lattice $\mathcal{L}(V)$ of all orthoclosed subspaces of V (recall that a linear subspace M of V is orthoclosed if $M = M^{\perp\perp}$, and $M^{\perp} = \{x \in V: \langle x, y \rangle = 0 \text{ for all } y \in M\}$). Owing to orthomodularity of L, the lattice $\mathcal{L}(V)$ satisfies the Hilbertian property $M + M^{\perp} = V$ for all $M \in \mathcal{L}(V)$ [where $+$ means taking the linear span; see Maeda and Maeda (1970)]. A four-tuple $(D, \theta, V, \langle ., . \rangle)$ with above properties is called an orthomodular space.

Moreover, L [and hence also $\mathcal{L}(V)$] is orthoisomorphic with the set $\mathcal{F}(P) := \{S \subset P: S = \overline{S}\}$ of all subsets of P which are closed under the formations of pure superpositions; meets in $\mathcal{F}(P)$ coincide with set-theoretic intersections, while suprema are defined by $\vee S_i = \overline{\cup S_i}$, and orthocomplementation is defined by $S^{\perp} = \{q \in P: q \perp s \text{ for all } s \in S\}$, where $p \perp q$ if the supports of p and q are orthogonal. A state p is a superposition of states q, r iff for the supports we have $a_p \leq a_q \vee a_r$.

It remains to specify the nature of the division ring D. We need axioms which make D one of the classical division rings R, C, or H. In Pulmannová (1994) two of Wilbur's (1977) axioms are used which make L a so-called probabilistic logic. These two axioms express minimal conditions which are necessary to consider vectors in V as sources of states. In the standard approach, the famous Gleason theorem asserts that such a situation naturally arises. In our situation, the atoms in $\mathcal{L}(V)$ are in one-to-one correspondence with the pure states in P. Therefore, the elements of V deserve consideration as possible sources of states. The following axiom, which is formulated independently on the representing space, guarantees that for each $x \in V$ there is a $d \in D$ with $\langle dx, dx \rangle = \pm 1$.

A7 (Normalizability). For each $f \in F := \{d \in D: \theta(d) = d\}$ there is a $d \in D$ with $f = \pm d\theta(d)$.

The second of Wilbur's axioms is justified by the fact that if $\langle x, x \rangle$ is to be a candidate for probability, then at least it should behave as a scalar and multiply commutatively in D. Its formulation is also independent of the underlying space V.

A8 (Scalar Behavior of Probabilities). F is a subset of the center C of D.

It was proved in Wilbur (1977) that if $(D, \theta, V, \langle ., .\rangle)$ is an infinite-dimensional orthomodular space satisfying A7 and A8, then D is one of R, C, or H.

Axioms A7 and A8 can be replaced by another axiom, using Solèr's (1995) theorem. Such an axiom, based on this very recent result, was formulated in Holland (1995). In Solèr's theorem, one may relax the assumption that $(D, \theta, V, \langle ., .\rangle)$ has an infinite orthonormal sequence to the assumption that V contains an orthogonal sequence $\{e_i: i \in \mathbb{N}\}$ of nonzero vectors such that $\langle e_i, e_i\rangle = \langle e_j, e_j\rangle$ for all i, j. If that is the case, then setting $f = \langle e_i, e_i\rangle$, we define a new involution γ on D by $\gamma(d) = f\theta(d)f^{-1}$ and a new form $[., .]$ on V by $[., .] = \langle ., .\rangle f^{-1}$. A direct calculation shows that the new form $[., .]$ is Hermitian with respect to the new involution γ and is also orthomodular because it induces the same $\perp$ map. The sequence $\{e_i\}$ is now orthonormal in $(D, \gamma, V, [., .])$, so by Solèr's theorem (D, γ) is R, C, or H. But $f = \gamma(f)$, so f is a nonzero real number. Thus $\gamma = \theta$, so (D, θ) is R, C, or H.

The weakened form of Solèr's theorem was used by Holland (1995) to formulate his Axiom D:

Axiom D (Ample Unitary Group). Given any two orthogonal pure states a, b of L, there is a unitary operator U such that $U(a) = b$.

Since Axiom A, Axiom B, and Axiom C in Holland (1995) are essentially the same as our Axioms A1–A6,[2] and lead to the same structure of L, namely that L is an irreducible, complete, atomic orthomodular lattice with covering property, and, moreover, pure states are in one-to-one correspondence with the atoms of L which are their supports, we can add Axiom D to our Axioms A1–A6 to arrive at a classical Hilbert space based on R, C, or H. Namely, by a unitary operator we mean a bijective linear map U of V into itself that preserves the form $\langle ., .\rangle$, $U(\alpha x + \beta y) = \alpha U(x) + \beta U(y)$, $\forall \alpha, \beta \in D$, $\forall x, y \in V$, $\langle U(x), U(y)\rangle = \langle x, y\rangle$ $\forall x, y \in V$. Now the pure states a, b are in one-to-one correspondence with atoms in $\mathcal{L}(V)$, and so $a = De$ and $b = Df$ for some orthogonal nonzero vectors $e, f \in V$, so for the operator U we shall

[2]Those axioms are formulated as follows: *Axiom A:* (1) The logic $\mathcal{L}$ is separable, i.e., any orthogonal family of nonzero elements in $\mathcal{L}$ is at most countable. (2) If $m(a) = m(b) = 0$ for some $a, b \in \mathcal{L}$ and $m \in \mathcal{S}$, then there exists $c \in \mathcal{L}$, $c \geq a$, $c \geq b$ with $m(c) = 0$. *Axiom B:* (1) Given any nonzero question $a \in \mathcal{L}$, there is a pure state $m \in \mathcal{S}$ with $m(a) = 1$. (2) If m is a pure state with support $a \in \mathcal{L}$, then m is the only state, pure or not, with $m(a) = 1$. *Axiom C:* (1) Given two different pure states (atoms) a and b, there is at least one other pure state c, $c \neq a$, $c \neq b$, that is a superposition of a and b. (2) If the pure state c is a superposition of the distinct pure states a and b, then a is a superposition of b and c.

have $U(e) = \alpha f$, for some $\alpha \in D$. Then $\langle e, e\rangle = \langle U(e), U(e)\rangle = \alpha\langle f, f\rangle\theta(\alpha)$. From this it follows that there exists in V an infinite orthogonal sequence $\{e_i: i \in \mathrm{N}\}$ such that $\langle e_i, e_i\rangle = \langle e_j, e_j\rangle$ for all i, j, and Solèr's theorem applies.

In quantum mechanics, based on the standard von Neumann (1955) approach complex Hilbertian lattices play a central role. Mayet and Pulmannová (1994) found a property which allows one to distinguish complex Hilbertian lattices among the classical ones. Based on this, a physically motivated axiom was suggested in Pulmannová (1994) which allows one to arrive at the lattice of closed subspaces of a complex Hilbert space.

A nearly orthosymmetric ortholattice (NOSOL) (Mayet and Pulmannová, 1994) is an ortholattice L equipped with a binary operation U satisfying the following axioms [where $U(a, b)$ is denoted by $U_a(b)$]:

(S1) For every $a \in L$, U_a is an automorphism of (L, U):
(a) $U_a(x^\perp) = U_a(x)^\perp$.
(b) $U_a(x \wedge y) = U_a(x) \wedge U_a(y)$.
(c) $U_a(1) = 1$.
(d) $U_a \circ U_b = U_{U_a(b)} \circ U_a$.
(S2) $x \vee U_a(b) = x \vee \phi_a(b)$, where $\phi_a(b) = a \wedge (a^\perp \vee b)$.
(S3) $a \perp b \Rightarrow U_a \circ U_b = U_{a \vee b}$.

If (L, U) is a NOSOL, then L is orthomodular. Moreover, if L is an atomistic ortholattice with covering property, then (S2) is isomorphic to

$$U_a(x) = x \quad \text{iff} \quad aCx$$

where aCx means that $a = (a \wedge x) \vee (a \wedge x^\perp)$ (Mayet and Pulmannová, 1994).

Let us consider an orthomodular space $(D, \theta, V, \langle.,.\rangle)$. Let $C(D)$ denote the center of D and $C_1(D) = \{\lambda \in C(D): \lambda\theta(\lambda) = 1\}$. Let $X \in \mathcal{L}(V)$, by the Hilbertian property, $\forall x \in V$, $x = x_1 + x_2$, $x_1 \in X$, $x_2 \in X^\perp$. For any $\lambda \in C_1(X)$, define a mapping $\sigma_{\lambda,X}: V \to V$, $\sigma_{\lambda,X}(x_1 + x_2) = x_1 + \lambda x_2$. The mapping $\sigma_{\lambda,X}$ is linear and preserves $\langle.,.\rangle$, hence $\sigma_{\lambda,X}(Y) = \{\sigma_{\lambda,X}(y): y \in Y\} \in \mathcal{L}(V)$ whenever $Y \in \mathcal{L}(V)$. Define a binary operation on $\mathcal{L}(V)$ by $U_\lambda(X, Y) + \sigma_{x,X}(Y)$. It has been proved in Mayet and Pulmannová (1994) that, for every $\lambda \in C_1(D)$, $\lambda \neq 1$, $(\mathcal{L}(V), U_\lambda)$ is a NOSOL. Moreover, if $\dim V \geq 3$ and $(\mathcal{L}(V), U)$ is a NOSOL, then there is a $\lambda \in C_1(D)$, $\lambda \neq 1$, such that $U = U_\lambda$.

Based on the above facts, the following characterization of complex Hilbert spaces can be formulated: Let p be any odd number ≥ 3 and let (L, U) be a NOSOL. Consider the equation

$$U_a^p = id_L, \qquad \forall a \in L \tag{C}$$

Let $L = \mathcal{L}(V)$, V over R, C, or H.

(a) If D = C, (C) holds for some NOSOL (L, U).
(b) If D = R or D = H, (C) does not hold in any NOSOL (L, U).

Indeed, according to Stone's theorem, any self-adjoint operator A on a complex Hilbert space H gives rise to a one-parameter strongly continuous unitary group $t \mapsto e^{iAt}$, $t \in$ R, and every such group is generated in this way by a self-adjoint operator. Let P be a projection and put $A = I - P$. We obtain

$$\begin{aligned} e^{it(I-P)} &= I + \sum_{n=1}^{\infty} \frac{(it(I - P))^n}{n!} \\ &= I + \sum_{n=1}^{\infty} \frac{(it)^n}{n!}(I - P) \\ &= I + (e^{it} - 1)(I - P) \end{aligned}$$

Define $U_{P,\lambda}x = e^{it(I-P)}x$, $x \in H$, $\lambda = e^{it}$. We obtain $U_{P,\lambda}x = x_1 + \lambda x_2$ whenever $x = x_1 + x_2$ with $x_1 \in P(H)$, $x_2 \in (I - P)(H)$. Accordingly, $e^{it(I-P)}$, for a suitable $t \in$ R, induces a NOSOL structure on $\mathcal{L}(H)$. Consider $e^{2\pi ri(I-P)}$, $r \in$ R. For $r = 1/n$, $n \in$ N, $n \neq 1$, we obtain a NOSOL. If $r = 1/p$, $p \geq 3$ is odd, (C) is satisfied.

On the other hand, in R and H, $C_1(D) = \{1, -1\}$, which entails that (C) cannot be satisfied.

Let $L = \mathcal{L}(V)$ be the logic of an infinite-dimensional orthomodular space. For $X \in L$ and $r \in$ R, let $r.X$ denote the observable such that $r.X(\{r\}) = X$, $r.X(\{0\}) = X^\perp$. The following axiom is suggested in Pulmannová (1994).

A9 (Weak Form of Stone's Theorem). To every observable $r.X$ on $\mathcal{L}(V)$ [$r \in$ R, $X \in \mathcal{L}(V)$] there exists an automorphism $U_X^{(r)}$ of $\mathcal{L}(V)$ satisfying the following conditions:

(i) $U_X^{(r)} = id$ iff either $X \in \{\{0\}, V\}$ ($r \in$ R is arbitrary) or r is an integer (X is arbitrary).
(ii) If $U_X^{(r)} \neq id$, then $U_X^{(r)}(Y) = Y$ iff XCY, $Y \in L$.
(iii) $X \perp Y \Rightarrow U_X^{(r)} \circ U_Y^{(r)} = U_{X+Y}^{(r)}$, $\forall r \in$ R.
(iv) $U_X^{(t)} \circ U_X^{(s)} = U_X^{(t+s)}$, $\forall t, s \in$ R, $\forall X \in L$.

It can be proved, using similar arguments as in the proof of Proposition 9 in Mayet and Pulmannová (1994) that if r is not an integer, then $U^{(r)}$

corresponds to U_λ for a suitable λ, and so $(L, U^{(r)})$ is a NOSOL. In particular, if $r = 1/p$ and $p \geq 3$ is odd, then (i) and (iv) imply that equation (C) is satisfied.[3]

In conclusion, Axioms A1–A8 (or, equivalently, Axioms A1–A6 and Axiom D) replace Mackey's Axiom VII, and we arrive at a logic based on a classical (real, complex, or quaternionic) Hilbert space. If in addition Axiom A9 is satisfied, we obtain the standard complex Hilbert space logic.

ACKNOWLEDGMENT

Supported by the grant of the Slovak Academy of Sciences No. 1229.

REFERENCES

Amemiya, I., and Araki, H. (1966/1967). A remark on Piron's paper, *Research Institute for Mathematical Sciences. Publication A*, **2**, 423–427.

Beltrametti, E., and Cassinelli, G. (1981). *The Logic of Quantum Mechanics*, Addison-Wesley, Reading, Massachusetts.

Birkhoff, G., and von Neumann, J. (1936). The logic of quantum mechanics, *Annals of Mathematics*, **37**, 823–834.

Bugajska, K., and Bugajski, S. (1972). On the axioms of quantum mechanics, *Bulletin de l'Academie Polonaise des Sciences, Series Mathematiques, Atronomiques, et Physiques*, **20**, 231–234 (1972).

Cirelli, A., and Cotta-Ramusino, P. (1973). On the isomorphism of a quantum logic with the logic of the projections in a Hilbert space, *International Journal of Theoretical Physics*, **8**, 11–19.

Dirac, P. (1980). *The Principles of Quantum Mechanics*, Clarendon Press, Oxford

Gleason, A. (1957). Measures on the closed subspaces of a Hilbert space, *Journal of Mathematics and Mechanics*, **6**, 885–894.

Gudder, S. P., and Piron, C. (1971). Observables and the field in quantum mechanics, *Journal of Mathematical Physics*, **12**, 1583–1588.

Holland, S. S., Jr. (1995). Orthomodularity in infinite dimensions: A theorem of M. Solèr, *Bulletin of the American Mathematical Society*, **32**, 205–234.

Jauch, J. (1968). *Foundations of Quantum Mechanics*, Addison-Wesley, Reading, Massachusetts.

[3] A short sketch of a proof is as follows. Let r be fixed and for any X put $U_X = U_X^{(r)}$. By Wigner's theorem (Piron, 1976) there exists a semilinear transformation σ on V which induces U_X, that is, for every $Y \in L$, $U_X(Y) = \{\sigma(y): y \in Y\}$. By (ii), the restrictions σ_X and $\sigma_{X^\perp}$ of σ to X and $X^\perp$, respectively, are semilinear transformations of X and $X^\perp$ which induce the identical maps respectively on $[0, X]$ and $[0, X^\perp]$ (observe that the interval $[0, X]$ in L can be identified with the lattice of all closed subspaces of X when X is considered as an orthomodular space). If neither X nor $X^\perp$ is one-dimensional, by Lemma 3 in Piron (1976), there are λ_1 and λ_2 in D such that $\sigma_X = \lambda_1\, id_X$, $\sigma_{X^\perp} = \lambda_2 id_{X^\perp}$. If X is one-dimensional, we choose a one-dimensional subspace Y orthogonal to X, and use that by (iii), $U_X \circ U_Y = U_{X \vee Y}$ to obtain that there are λ_1 and λ_2 as above. Therefore in any case if $x \in V$, $x = x_1 + x_2$, $x_1 \in X$, $x_2 \in X^\perp$, $\sigma(x_1 + x_2) = \sigma_X(x_1) + \sigma_{X^\perp}(x_2) = \lambda_1 x_1 + \lambda_2 x_2$. Since σ is defined up to a multiplicative constant, we can write $\sigma(x_1 + x_2) = x_1 + \lambda x_2$. The proof continues exactly as the proof of Proposition 9 in Mayet and Pulmannová (1994).

Jauch, J., and Piron, C. (1969). On the structure of quantal proposition systems, *Helvetica Physica Acta*, **42**, 842–848.
Keller, H. A. (1980). Ein nichtclassicher Hilbertscher Raum, *Mathematische Zeitschrift*, **172**, 41–49.
Mackey, G. (1963). *The Mathematical Foundations of Quantum Mechanics*, Benjamin, New York.
MacLaren, M. (1964). Atomic orthocomplemented lattices, *Pacific Journal of Mathematics*, **14**, 597–612.
Maczyński, M. (1973). The field of real numbers in axiomatic quantum mechanics, *Journal of Mathematical Physics*, **14**, 1469–1471.
Maeda, F., and Maeda, S. (1970). *Theory of Symmetric Lattices*, Springer-Verlag, Berlin, Heidelberg, New York.
Mayet, R., and Pulmannová, S. (1994). Nearly orthosymmetric ortholattices, *Foundations of Physics*, **24**, 1425–1437.
Piron, C. (1964). Axiomatique quantique, *Helvetica Physica Acta*, **37**, 439–468.
Piron, C. (1976). *Foundations of Quantum Physics*, Benjamin, Reading, Massachusetts.
Pták, P., and Pulmannová, S. (1991). *Orthomodular Structures as Quantum Logics*, Kluwer, Dordrecht.
Pulmannová, S. (1994). Quantum logics and Hilbert space, *Foundations of Physics*, **24**, 1403–1414.
Rüttimann, G. (1977). Jauch–Piron states, *Journal of Mathematical Physics*, **18**, 189–193.
Solèr, M. P. (1995). Characterization of Hilbert spaces by orthomodular spaces, *Communications in Algebra*, **23**(1), 219–243.
Varadarajan, V. (1968). *Geometry of Quantum Theory I*, van Nostrand, Princeton, New Jersey.
Von Neumann, J. (1955). *Mathematical Foundations of Quantum Mechanics*, Princeton University Press, Princeton, New Jersey.
Wilbur, W. (1977). On characterizing the standard quantum logics, *Transactions of the American Mathematical Society*, **233**, 265–282.
Zierler, N. (1961). Axioms for non-relativistic quantum mechanics, *Pacific Journal of Mathematics*, **11**, 1151–1169.
Zierler, N. (1966). On the lattice of closed subspaces of Hilbert space, *Pacific Journal of Mathematics*, **19**, 583–586.